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Quadrature Formulas for Semi-Infinite Integrals*

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Abstract. Polynomials orthogonal on $[0, \infty]$ with regard to the weight function $w(x) = x^{\alpha}(1 + x)^{-\beta}$ are obtained, recurrence relations are found and the differential equation, which is satisfied by them, is given. Formulas for weights and abscissas in the corresponding quadrature formula are given.

1. Introduction. Harper [1] developed a quadrature formula on $[-\infty, \infty]$ with the algebraic weight function $w(x) = (1 + x^2)^{-k-1}$. Even though the degree of the polynomial approximation to $f(x)$ is limited, he illustrated the superiority of his formula over that due to Gauss-Hermite, when $f(x)$ is a particular algebraic function.

In this paper we have worked out a similar formula on $[0, \infty]$ with the algebraic weight function $w(x) = x^{\alpha}(1 + x)^{-\beta}$. While the Gauss-Laguerre formula is expected to work well in situations where the integrand function behaves asymptotically like $e^{-\gamma x}$, our formula would be more suitable where the integrand function has both the following features:

(a) asymptotic behaviour like $x^{-\beta+n}$,

(b) algebraic singularity like $(1 + x)^{-\beta}$ at $x = -1$.

This fact has been illustrated with the help of an example in Section 3.

2. Derivation of Formulas. The existence of the formula

$$\int_0^{\infty} x^{\alpha}(1 + x)^{-\beta} f(x)\, dx = \sum_{i=1}^{n} H_i f(x_i) + E_{n,\alpha,\beta} \tag{1}$$

requires that

$$\int_0^{\infty} x^{\alpha}(1 + x)^{-\beta}\, dx \geqq 0 \tag{2}$$

and that the growth of the function $f(x)$, as a approaches infinity, be restricted by the condition

$$|f(x)| \leqq M \cdot |x|^p \tag{3}$$

where M is a finite constant and p is a real number such that $p + \alpha < \beta$.

In (1) the abscissas x_i are the zeros of the nth degree polynomial $\phi_{n,\alpha,\beta}(x)$ which satisfies the orthogonality condition

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$$\int_0^\infty x^\alpha(1+x)^{-\beta}\phi_{m,\alpha,\beta}(x)\phi_{n,\alpha,\beta}(x)\,dx = 0, \qquad m \neq n. \tag{4}$$

Following Hildebrand [2] it is found that such a system of orthogonal polynomials is given by the formula

$$\phi_{n,\alpha,\beta}(x) = c_n x^{-\alpha}(1+x)^\beta \frac{d^n}{dx^n}[x^{n+\alpha}(1+x)^{n-\beta}].$$

If these polynomials are standardised such that the coefficient of x^n is unity, we require

$$c_n = (-1)^n\Gamma(\beta-2n-\alpha)/\Gamma(\beta-n-\alpha).$$

Thus, the Rodrigues formula for $\phi_{n,\alpha,\beta}(x)$ is

$$\phi_{n,\alpha,\beta}(x) = (-1)^n \frac{\Gamma(\beta-2n-\alpha)}{\Gamma(\beta-n-\alpha)} x^{-\alpha}(1+x)^\beta \frac{d^n}{dx^n}[x^{n+\alpha}(1+x)^{n-\beta}] \tag{5}$$

$$[\alpha > 0, \beta > \alpha+1, n < \tfrac{1}{2}(\beta-\alpha)].$$

Here α, β are any real numbers such that $\alpha > 0$, $\beta > \alpha + 1$ and n is the greatest integer less than $\frac{1}{2}(\beta-\alpha)$. These restrictions are imposed by condition (2) and by the requirement for the existence of orthogonal polynomials.

The following recurrence relations are found to be satisfied by (5):

$$\begin{aligned}\phi_{n,\alpha,\beta}(x) &= \frac{1}{2}\left[(2x+1) - \frac{\beta^2-\alpha^2}{(\beta-2n-\alpha)(\beta-2n-\alpha+2)}\right]\phi_{n-1,\alpha,\beta}(x) \\ &\quad - \frac{(n-1)(\beta-n+1)(n+\alpha-1)(\beta-n-\alpha+1)}{(\beta-2n-\alpha+3)(\beta-2n-\alpha+2)^2(\beta-2n-\alpha+1)}\phi_{n-2,\alpha,\beta}(x),\end{aligned} \tag{6}$$

$$n = 2, 3, 4, \cdots,$$

$$\phi_{0,\alpha,\beta}(x) = 1, \qquad \phi_{1,\alpha,\beta}(x) = x - (\alpha+1)/(\beta-\alpha-2),$$

$$\begin{aligned}x(1+x)\phi'_{n,\alpha,\beta}(x) + (\beta-\alpha-n-1)\left[\frac{n+\alpha+1}{\beta-2n-\alpha-2} - x\right]\phi_{n,\alpha,\beta}(x) \\ + (\beta-2n-\alpha-1)\phi_{n+1,\alpha,\beta}(x) = 0,\end{aligned} \tag{7}$$

$$\begin{aligned}x(1+x)\phi'_{n,\alpha,\beta}(x) &= n\left[\frac{\beta-n}{\beta-2n-\alpha} + x\right]\phi_{n,\alpha,\beta}(x) \\ &\quad + \frac{(\beta-n)(n+\alpha)n(\beta-n-\alpha)}{(\beta-2n-\alpha)^2(\beta-2n-\alpha+1)}\phi_{n-1,\alpha,\beta}(x),\end{aligned} \tag{8}$$

$$\begin{aligned}&\phi_{n,\alpha,\beta-1}(x) \\ &= \frac{(\beta-n-2)(\beta-n-\alpha-1)\phi_{n,\alpha,\beta}(x) + (\beta-2n-\alpha-1)(\beta-2n-\alpha-2)\phi_{n+1,\alpha,\beta}(x)}{(1+x)(\beta-2n-\alpha-1)(\beta-2n-\alpha-3)},\end{aligned} \tag{9}$$

$$\begin{aligned}&\phi_{n,\alpha+1,\beta}(x) \\ &= \frac{1}{x}\left[\frac{(n+\alpha+1)(\beta-n-\alpha+1)}{(\beta-2n-\alpha-1)(\beta-2n-\alpha-2)}\phi_{n,\alpha,\beta}(x) + \phi_{n+1,\alpha,\beta}(x)\right].\end{aligned} \tag{10}$$

The differential equation satisfied by the polynomial $\phi_{n,\alpha,\beta}(x)$ is

$$x(1+x)y'' + [1+\alpha-(\beta-\alpha-2)x]y' + n(\beta-\alpha-n-1)y = 0. \tag{11}$$

The polynomials (5) are related to the Jacobi polynomials $P_n^{(a,b)}(x)$ and to the hypergeometric functions $F(a, b; c; x)$ as follows:

$$\phi_{n,\alpha,\beta}(x) = n!\,\frac{\Gamma(\beta-2n-\alpha)}{\Gamma(\beta-n-\alpha)}\,P_n^{(-\beta,\alpha)}[-(2x+1)], \tag{12}$$

$$\phi_{n,\alpha,\beta}(x) = \frac{\Gamma(\beta)\Gamma(\beta-2n-\alpha)}{\Gamma(\beta-n)\Gamma(\beta-n-\alpha)}\,x^n\cdot F\left(-n, -n-\alpha; -\beta+1; \frac{x+1}{x}\right). \tag{13}$$

The weight coefficients and the error term in (1) are computed to be given by

$$H_i = n!\,\Gamma(n+\alpha+1)(\Gamma(\beta-2n-\alpha))^2[x_i(1+x_i)]^{-1}[\phi'_{n,\alpha,\beta}(x_i)]^{-2}, \tag{14}$$

$$E_{n,\alpha,\beta} = \frac{f^{(2n)}(\xi)}{(2n)!}\int_0^\infty x^\alpha(1+x)^{-\beta}[\phi_{n,\alpha,\beta}(x)]^2\,dx$$

$$= \frac{n!\,\Gamma(n+\alpha+1)(\Gamma(\beta-2n-\alpha))^2}{\Gamma(\beta-n-\alpha)\Gamma(\beta-n)(\beta-2n-\alpha-1)(2n)!}\,f^{(2n)}(\xi) \tag{15}$$

$$[n < \tfrac{1}{2}(\beta-\alpha) - \tfrac{1}{2},\ 0 < \xi < \infty].$$

3. Illustration. For $\alpha = 1$, and $\beta = 13$, the following set of polynomials is generated:

$$\phi_0 = 1,$$
$$\phi_1 = x - \tfrac{1}{5},$$
$$\phi_2 = x^2 - \tfrac{3}{4}x + \tfrac{1}{12},$$
$$\phi_3 = x^3 - 2x^2 + \tfrac{6}{7}x + \tfrac{1}{14},$$
$$\phi_4 = x^4 - 5x^3 + 6x^2 - 2x + \tfrac{1}{7},$$
$$\phi_5 = x^5 - 15x^4 + 50x^3 - 50x^2 + 15x - 1.$$

For the quadrature formula

$$\int_0^\infty \frac{x}{(1+x)^{13}}\,f(x)\,dx = \sum_{i=1}^n H_i f(x_i), \tag{16}$$

the values of x_i and H_i are given in Table 1.

As an illustration, the integral

$$I(f) = \int_0^\infty \frac{x(1+x)^{1/2}}{(1+x)^{13}}\,dx = 0.0082\ 8157 \tag{17}$$

has been approximated using formula (16) and also using a straightforward application of the Gauss-Laguerre formula. This example is 'hand-tailored' for the use of formula (16) and, since it does not behave asymptotically like e^{-x}, the Gauss-Laguerre formula is quite inappropriate. The numerical results listed in Table 2 show that for this example, formula (16) is much more accurate than the Gauss-Laguerre formula.

TABLE 1
Abscissas and Weights for Quadrature (17)

x	x_i	H_i
1	0.2000 0000	(−2) 0.7575 7575
2	0.6143 5678	(−2) 0.1018 4615
	0.1356 4322	(−2) 0.6557 2960
3	0.1100 2615	(−2) 0.5645 9170
	0.4512 2182	(−2) 0.1906 3812
	(1) 0.1438 7520	(−4) 0.2345 9366
4	(−1) 0.9786 2600	(−3) 0.1039 5084
	0.3868 4283	(−4) 0.4906 0878
	(1) 0.1107 2489	(−5) 0.1594 2659
	(1) 0.3408 0457	(−8) 0.1309 1096
5	(−1) 0.9276 2478	(−2) 0.3982 9769
	0.3616 1054	(−2) 0.1066 6655
	1.0000 0000	(−4) 0.2976 1905
	(1) 0.2765 4061	(−6) 0.8624 1404
	(2) 0.1078 0221	(−9) 0.2354 0418

TABLE 2
Error in the Evaluation of (17)

n	Gauss-Laguerre	Formula (16)	Upper bound (15)
1		-1725×10^{-8}	-2525×10^{-8}
2	$54\ 6101 \times 10^{-8}$	-34×10^{-8}	-129×10^{-8}
3	$24\ 7277 \times 10^{-8}$	-2×10^{-8}	-58×10^{-8}
4	$1\ 1412 \times 10^{-8}$		
5	$-13\ 6969 \times 10^{-8}$		
6	$-21\ 4710 \times 10^{-8}$		
7	$-24\ 4831 \times 10^{-8}$		
8	$-24\ 5982 \times 10^{-8}$		
16	$-7\ 8818 \times 10^{-8}$		
24	$-1\ 6744 \times 10^{-8}$		
32	$-\ 3506 \times 10^{-8}$		

The last column of Table 2 contains the upper bounds on errors calculated through the estimate (15) after replacing $f^{(2n)}(\xi)$ by $\max_{0 \leq x \leq \infty} |f^{(2n)}(x)|$.

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